

# Permutations & Combinations

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## 1 Permutations

Given a set of objects we may consider how many ways there are of writing that set in order.

Alternately, we may ask how many ways we may rearrange the elements of a set.

If the set has  $n$  elements there are  $n$  choices for the first position,  $n - 1$  choices for the second position, and so on. When we reach the last position we have only one element left.

So by the multiplication rule the number of *permutations* of an  $n$  set is

$$n! = n(n - 1)(n - 2) \dots 3 \cdot 2 \cdot 1$$

### Example 1

1. How many ways are there to write out the integers 1,2,3,4

$$1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, \dots$$

For a total of  $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$  possibilities.

2. What is the probability that a random 4 digit number contains exactly the integers 1 through 4?

$$\frac{24}{9999 - 1000 + 1} = \frac{24}{9000} = \frac{1}{375} \approx 0.00267$$

We also want to consider cases when there are more elements than slots.

Another way of looking at this is that we are sampling from a set without replacement.

### Example 2

Let  $S = \{a, b, c, d, e, f, g\}$  How many 3 character strings are there over  $S$  which do not repeat a character?

Note that  $|S| = 7$ .

Thus there are 7 choices for the first character. This leaves 6 characters left.

We now have 6 choices for the second character. This leaves 5 characters left.

We now have 5 choices for the last character.

Thus there are  $7 \cdot 6 \cdot 5 = 210 = \frac{7!}{(7-3)!}$  three character strings over  $S$  which do not repeat a character.

**Theorem 3 ( $r$ -permutations of  $n$  Objects)** *The number of ways of choosing  $r$  times from an  $n$  set is denoted  $P(n, r)$  and*

$$P(n, r) = \frac{n!}{(n - r)!}$$

### Notes

- We must have  $n \geq r$ .

- If  $n = r$   $P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ .
- The choice is made without replacement.
- The order in which we choose matters.

**Example 4**

1. How many different ways are there of reordering the letters of the word TIMER?

$$5! = 120$$

How many of these have R as the first letter?

We are now essentially placing 4 letters into 4 positions:

$$4! = 24.$$

2. How many ways can we choose 3 of the letters of the word SPRING?

$$P(6, 3) = 6 \cdot 5 \cdot 4 = 120.$$

We now consider the case when 2 or more of the elements are indistinguishable.

For example, how many different ways are there of reordering the letters of the word PAPER?

First we distinguish the two Ps:  $P_1AP_2ER$  and there are  $5!$  permutations. However, the permutations  $AP_1EP_2R$  and  $AP_2EP_1R$  are the same. For every arrangement there will be another with the order of the P's interchanged. So the true number is  $\frac{5!}{2} = 60$ .

**Theorem 5** Suppose  $S$  contains  $n$  objects of  $k$  different types, such that there are  $n_1$  objects of type 1,  $n_2$  objects of type 2, etcetera up to  $n_k$  objects of type  $k$ . If  $P(n; n_1, n_2, \dots, n_m)$  represents the number of distinct permutations of  $S$ , then

$$P(n; n_1, n_2, \dots, n_m) = \frac{n!}{(n_1!)(n_2!) \dots (n_k!)}$$

1. How many different ways are there of reordering the letters of the word REELED?

As above the 3 E's are indistinguishable. For each of the  $6!$  permutations of the six letters, there are  $3!$  ways to place the E's. So the total is  $\frac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120$

2. How many different ways are there of reordering the letters of the word REORDER?

The three R's are indistinguishable and (independantly) the two Es are indistinguishable.

$$\frac{7!}{3!2!} = 420.$$

## 2 Combinations

In the case of permutations the order in which the elements are drawn is important. But in many applications the order in which the objects are drawn is immaterial, only the elements themselves matter.

For example if we are dealt cards from a deck to form a hand, the order in which the cards are dealt doesn't matter, only the cards in the hand are of consequence.

If we draw  $r$  objects from an  $n$  set, they may arrive in any of  $r!$  ways.

**Theorem 6 ( $r$ -combinations of  $n$  Objects)** *The number of ways of choosing  $r$  times from an  $n$  set without order is denoted  $C(n, r)$ , or  $\binom{n}{r}$  and*

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

### Notes

- $\binom{n}{r}$  is read “ $n$  choose  $r$ ”.
- We must have  $n \geq r$ .
- If  $n = r$ ,  $\binom{n}{n} = \frac{n!}{(n-n)!n!} = 1$ .
- The choice is made without replacement.
- The order in which we choose does not matter.

### Example 7

1. How many ways can we choose a 3 set from a 10 set?

Note that sets do not have order, so we should use unordered choice; combinations

$$C(10, 3) = \frac{10!}{7!3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = \frac{720}{6} = 120.$$

2. If 5 cards are drawn from a standard deck, what is the probability that all of them will be spades?

The total number of 5 card hands is  $\binom{52}{5} = \frac{52!}{47!5!} = 2,598,960$ .

There are 13 spades, the number of ways of choosing 5 all spades is  $\binom{13}{5} = \frac{13!}{8!5!} = 1,287$ .

$$P(5\spadesuit) = \frac{1287}{2598960} = 0.00049519807923 \approx 5 \times 10^{-4} = \frac{1}{2,000}.$$

3. Find the mistake in the following argument:

We are drawing 5 cards, the probability of drawing a spade is  $P(\spadesuit) = \frac{13}{52} = \frac{1}{4}$ . Thus the probability of 5 spades is  $\left(\frac{1}{4}\right)^5 = 0.0009765625$

The problem is that this assumes sampling with replacement. Given that a spade was drawn the first time the probability of a spade being drawn is now  $\frac{12}{51}$ , not  $\frac{1}{4}$

### 3 Choosing with Repetition

We now consider the problem of choosing with repetition allowed.

Suppose that we wish to know how many ways we can select  $r$  objects from an  $n$  set with repetition.

Let  $n_i$  be the number of times object  $a_i$  is chosen  $1 \leq i \leq n$ . Then  $n_1 + n_2 + \dots + n_n = r$ .

We can represent this situation by dividing the  $n$  possibilities with a  $|$  and placing a  $\times$  in the  $i$ th box for each occurrence of  $a_i$ . For example

$$\times \times | | \times | \times$$

would mean that 2 of object 1 were chosen ( $n_1 = 2$ ), none of object 2 ( $n_2 = 0$ ) and one of each of objects 3 and 4 ( $n_3 = n_4 = 1$ ).

There are  $n - 1$   $|$ 's and  $r$   $\times$ 's. The number of choices of  $\times$ 's and  $|$ 's is  $\binom{n+r-1}{r}$ .

**Theorem 8** *The number of  $r$ -combinations of an  $n$  set with repetition allowed is*

$$\binom{n+r-1}{r}$$

#### Example 9

How many ways are there to choose a 4-set from a 6-set?

Let  $S = \{1, 2, 3, 4, 5, 6\}$  be our 6-set.

We are choosing from 1 to 6 four times with repetition. We must place 4  $\times$ 's and  $6 - 1 = 5$   $|$ 's.

$$\binom{6+4-1}{4} = \binom{9}{4} = \frac{9!}{5!4!} = 126.$$

Suppose that we have  $n$  unknowns  $x_1$  to  $x_n$ ,  $x_i \in \mathbb{N}$  and  $x_1 + x_2 + \dots + x_n = r$ , we can use exactly the same argument as above to calculate how many Natural numbers satisfy this equation.  $\binom{n+r-1}{r}$

#### Example 10

How many Natural number solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

$n = 4$ ,  $r = 15$ , so

$$\binom{18}{15} = \frac{18!}{15!3!} = \frac{18 \cdot 17 \cdot 16}{6} = 3 \cdot 17 \cdot 16 = 816$$

Note that for this application we can have  $r > n$ .

## 4 The Algebra of Combinations

Combinations satisfy many interesting and useful relationships. We will investigate a few of them here.

**Theorem 11**

$$\binom{n}{r} = \binom{n}{n-r}$$

**Proof:**

$$\begin{aligned} \binom{n}{r} &= \frac{n!}{(n-r)!r!} && \text{(Definition of } \binom{n}{r} \text{)} \\ &= \frac{n!}{(n-r)!(n-(n-r))!} && \text{(Algebra - } r = n - (n-r) \text{)} \\ &= \binom{n}{n-r} \end{aligned}$$

Another way of looking at this is that the number of ways of choosing  $r$  objects from  $n$  things is the same as the number of ways of not choosing the remaining  $n - r$  objects.

We have the following

$$\begin{aligned} \binom{n}{1} &= \binom{n}{n-1} = n \\ \binom{n}{2} &= \binom{n}{n-2} = \frac{1}{2}n(n-1) \end{aligned}$$

**Theorem 12 (Pascals Formula)**

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

**Proof:** Let  $n$  and  $r$  be positive integers with  $n \geq r$ .

Let  $S = \{x_1, x_2, \dots, x_n, x_{n+1}\}$  be an  $n + 1$  set.

We enumerate all  $\binom{n+1}{r}$   $r$  subsets of  $S$ .

The number of  $r$  subsets of  $S$  which do not contain  $x_{n+1}$  is  $\binom{n}{r}$ .

The number of  $r$  subsets which contain  $x_{n+1}$  is  $\binom{n}{r-1}$ .

The result follows from the addition rule.

This may be used to calculate new combinations from old.

Table of  $\binom{n}{r}$

	0	1	2	3	...	$r-1$	$r$
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		
$n$	$\binom{n}{0} = 1$	$\binom{n}{1} = n$	$\binom{n}{2}$	$\binom{n}{3}$	...	$\binom{n}{r-1}$	$\binom{n}{r}$
$n+1$	1	$n+1$	$\binom{n+1}{2} = n + \binom{n}{2}$	$\binom{n+1}{3} = \binom{n}{2} + \binom{n}{3}$	...	$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$	

**Theorem 13 (The Binomial Theorem)**

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i$$

**Proof:** Let  $x, y \in \mathbb{R}$  and  $n \in \mathbb{Z}^+$ .

Consider how many times a terms of the form  $x^{n-1}y^i$  occurs in the expansion of  $(x + y)^n$ . For each way of choosing  $i$   $y$ 's there will be a term.

**Theorem 14**

$$\sum_{i=0}^n \binom{n}{i} = 2^n$$

**Proof:**  $2^n = (1 + 1)^n = \sum_{i=0}^n \binom{n}{i}$  by the Binomial Theorem.