RYERSON UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE

CPS 420
FINAL EXAM
WINTER 2017

NAME: _____

STUDENT ID: _____

INSTRUCTIONS

- This exam is 3 hours long.

- This exam is out of 80 and is worth 40% of the course mark.

- This is a closed book exam. However, one double-sided letter-sized crib sheet is allowed. No other aids are allowed.

- This exam is single-sided and has 8 pages including this front page.

- Please answer all questions directly on this exam.

- The exam is divided into 4 parts ordered chronologically as the material was covered in the course. You might find it helpful to read the whole exam and start with the sections you find easiest.

For Grading Purposes

A1-2	**/10**
A3	**/10**
B	**/10**
C	**/20**
D1-2	**/12**
D3	**/11**
D4	**/7**

PART A – INDUCTION AND RECURSION – 20 MARKS

Given the sequence a_n defined recursively as follows:

$a_0 = 1$

$a_n = 3a_{n-1}+2$ for $n \geq 1$

A1   Terms of a Sequence (5 marks)

Calculate a_1, a_2, a_3, a_4, a_5
Keep your intermediate answers as you will need them in the next question.

A2   Iteration (5 marks)

Using iteration, solve the recurrence relation when $n \geq 1$ (i.e. find an analytic formula for a_n).
Simplify your answer as much as possible, showing your work and quoting any formula or rule
that you use. In particular, your final answer should not contain Σ sums and Π products.

You can use this formula without proving it: for any positive integers b and n with $b \neq 1$,
$\sum_{i=0}^{n} b^i = \frac{b^{n+1} - 1}{b - 1}$

A3   Induction (10 marks)

Given the sequence b_n defined recursively as follows:

$b_0 = 1$

$b_n = \dfrac{b_{n-1}}{1+2b_{n-1}}$ for n ≥ 1

Prove by weak induction that for every n≥0, $b_n = \dfrac{1}{2n+1}$

No other method is acceptable.

Be sure to lay out your proof clearly and correctly and to justify every step.

PART B - GRAPH THEORY – 10 MARKS

B1   Drawing Graphs (10 marks)

In each of the following questions, either draw a graph with the given specifications or explain why no such graph exists.

    a) Build a graph which is not connected and had 6 vertices and 12 edges

    b) Build a graph which is connected, has 6 vertices, 5 edges, and a circuit.

    c) Build a tree with 6 vertices and 6 edges:

    d) Build a forest with 8 vertices and 5 edges

    e) Build a graph with 6 vertices and 5 edges which is not a tree.

PART C – REGULAR EXPRESSIONS AND FINITE STATE AUTOMATA – 20 MARKS

Let L_1 be the language of the **binary representations** of all **positive** integers divisible by 4.
Let L_2 be the language of the **binary representations** of all **positive** integers **not** divisible by 4.
None of the elements of these languages have leading zeroes.

Please note that all the questions in this part of the exam are related. They have been ordered in an order that facilitates answering them but it is possible to answer them independently. Therefore you can answer these questions in the order that makes the most sense to you.

C1   Regular Expressions   (6 marks)

     a)   Write a regular expression denoting L_1.

     b)   Write a regular expression denoting L_2.

C2   Finite State Automata   (14 marks)

     a)   Draw a state diagram (= deterministic finite state automaton) with as few states as possible which recognizes L_1. This state diagram should be complete: it should handle all strings of 0's and 1's.

     b)   Draw a state diagram (= deterministic finite state automaton) with as few states as possible which recognizes L_2. This state diagram should be complete: it should handle all strings of 0's and 1's.

PART D – COUNTING AND PROBABILITIES – 30 MARKS

In this entire section, you should simplify your calculations as much as possible

D1   Final Exam Grades   (6 marks)

There are 168 students currently registered in CPS420. This final exam is graded out of 80. For the purposes of this question you can assume that all the marks are integers.

a)   What is the **least** number of final exams that will need to be graded to guarantee that at least 2 students in this class have the same grade in this final exam?   Explain your answer.

b)   You want to make a bet that there will be a group of at least x CPS420 students who will all have the same grade on this exam. How large can you make x and still be **guaranteed** to win your bet? Explain your answer.

D2   Smart Hat   (6 marks)

A hat contains 25 tokens of identical size, shape and weight. The tokens are numbered from 1 to 25. Your friend draws a token randomly from the hat and tells you that its number has two digits. What is the probability that your friend drew a token with a prime number? Explain your answer.

Note that the set of all prime numbers between 1 and 25 is {2, 3, 5, 7, 11, 13, 17, 19, 23}

D3   Anagrams   (11 marks)

An anagram of a word is a new word that is formed by rearranging the letters of the original word. For example EVIL is an anagram of VILE.

   a)   How many different anagrams of the word KEEPER are there? (including KEEPER)
           Explain your answer.

   b)   How many of these begin with E or P?
           Explain your answer.

   c)   What is the probability that a random anagram of KEEPER (including KEEPER) will
           start with E or P? Explain your answer.

<u>D4</u>   <u>Binomial Theorem</u>   <u>(7 marks)</u>

a) Finish drawing the first 5 lines of Pascal's triangle by filling in the circles below with the missing values:
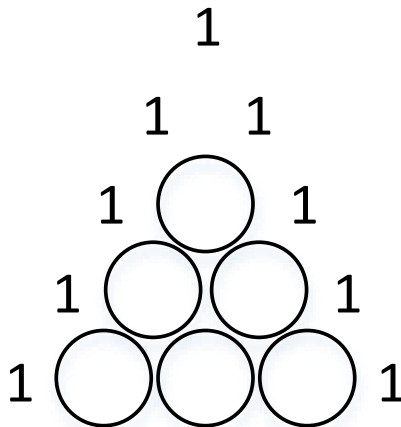
b) Use the drawing in part a) to show a full expansion of $(2x+5)^4$. **Show your intermediate steps.**